Note 1

§ Complex Numbers

Motivation: Solve eq. $x^2 = -1$. no real solutions, "square root of -1"

Solution: Symbol i pure imaginary number satisfies $i^2 = -1$.

- Solve general quadratic equation: e.g. $z^2 + z + 1 = 0$

$$\Rightarrow z = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$

- Fundamental Theorem of Algebra: Every degree n polynomial has exactly n complex roots.

Def: Complex number $\mathbb{C}$ $\quad z = x + yi, \quad x, y \in \mathbb{R}$

$\operatorname{Re} z$ $\quad \operatorname{Im} z$ $\quad$ Real & imaginary part

- Addition & Subtraction: $\quad z_1 = x_1 + y_1 i, \quad z_2 = x_2 + y_2 i$

$$z_1 \pm z_2 = (x_1 \pm x_2) + (y_1 \pm y_2) i$$

- Multiplication: $\quad (x_1 + y_1 i) \cdot (x_2 + y_2 i)$

$$= x_1 x_2 + x_1 y_2 i + x_2 y_1 i + y_1 y_2 i^2$$

$$= (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i$$

No need to remember formula, Just apply distributive law.

e.g. $z = x + yi$ $\bar{z} = x - yi$ Complex Conjugate

$$z \cdot \bar{z} = x^2 + y^2$$

$$\Rightarrow \underbrace{(x+yi)}_{z} \cdot \underbrace{\left(\frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} i \right)}_{z^{-1} = \frac{\bar{z}}{x^2+y^2}} = 1$$

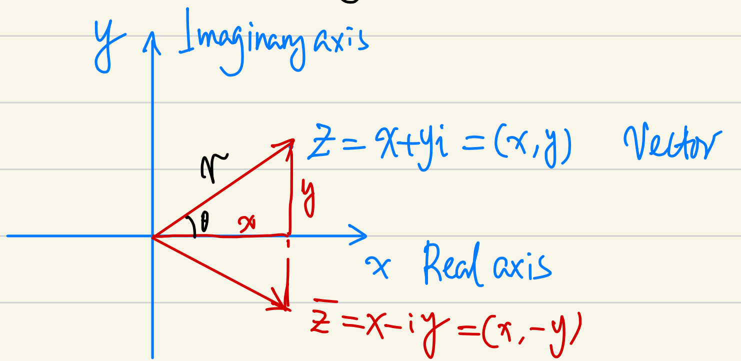
inverse

- Division: $\dfrac{z_1}{z_2} = z_1 \cdot z_2^{-1}$

e.g. $\dfrac{4+i}{2-3i} = (4+i) \cdot (2-3i)^{-1} = (4+i) \cdot \dfrac{2+3i}{2^2+(-3)^2} = \dfrac{5+14i}{13}$

equivalently, $= \dfrac{4+i}{2-3i} \cdot \dfrac{2+3i}{2+3i} = \dfrac{(4+i)(2+3i)}{13}$.

§ Complex plane - geometry of $\mathbb{C}$.

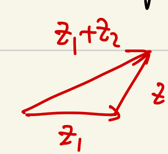


- Complex addition / subtraction $\leftrightsquigarrow$ Vector addition / subtraction

- Complex conjugate $\leftrightsquigarrow$ reflection along x-axis

- Modulus / absolute value / norm / magnitude of z: $|z| = \sqrt{x^2 + y^2}$ ← length of vectors

$\Rightarrow$ Satisfying triangle inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$



· Polar Coordinates (r, θ) : r: $|z| =$ modulus $=$ length ...

θ: $\arg(z) =$ argument of z

Polar form

Rectangle v.s Polar: $x = r\cdot\cos\theta$

$y = r\cdot\sin\theta$.

$$\boxed{\Rightarrow \quad z = x + yi = r(\cos\theta + i\cdot\sin\theta)}$$

Rmk: · θ not uniquely defined: Can add a multiple of 2π.

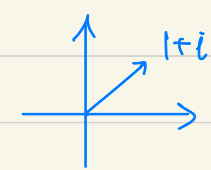
· Unique value if fixing $-\pi < \theta \leq \pi$, denoted $\Theta = \mathrm{Arg}(z)$

$\arg z = \mathrm{Arg}(z) + 2k\pi$

Sometimes $0 \leq \theta < 2\pi$

e.g. $z = 1+i$. $r = \sqrt{2}$ $\theta = \frac{\pi}{4}$, $\frac{\pi}{4} \pm 2\pi$, $\frac{\pi}{4} \pm 4\pi$, ...

$\Theta = \frac{\pi}{4}$



- **Euler's formula** : $\boxed{e^{i\theta} = \cos\theta + i \cdot \sin\theta}$ ← take as definition / notation temporarily.

$$z = x + yi$$
$$= r(\cos\theta + i\sin\theta)$$
$$= r \cdot e^{i\theta}$$ ← Exponential form

e.g. $z = 1 + i$
$= \sqrt{2}\, e^{i \cdot \frac{\pi}{4}}$

What's it good for? Multiplication!

Recall: $e^a \cdot e^b = e^{a+b}$ for $a, b \in \mathbb{R}$.

$$e^{i\theta_1} \cdot e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= (\underbrace{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}_{\cos(\theta_1+\theta_2)}) + i(\underbrace{\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1}_{\sin(\theta_1+\theta_2)})$$

$$= e^{i(\theta_1+\theta_2)}$$

In general. $z_1 = r_1 e^{i\theta_1}$ $\quad z_2 = r_2 e^{i\theta_2}$

then $\boxed{z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}}$

$\Rightarrow$ · Power: $\quad z^n = r^n \cdot e^{in\cdot\theta}$

ex: $(1+i)^6 = (\sqrt{2} e^{i\frac{\pi}{4}})^6 = 8 \cdot e^{i\frac{6\pi}{4}} = -8i$

· Inverse $\quad \frac{1}{z} = \frac{1}{r} \cdot e^{-i\theta}$

· Quotient: $\quad \frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot e^{i(\theta_1-\theta_2)}$

Application: Roots of complex number

Solve eq. $z^n = c \qquad c \in \mathbb{C}$.

Write $c = Re^{i\phi}$, $z = re^{i\theta}$ exponential form

$$z^n = r^n e^{in\theta} = Re^{i \cdot \phi}.$$

$\Rightarrow \quad r = R^{1/n}$; $n\theta = \phi + 2\pi \cdot k$ where $k = 0, \pm 1, \pm 2, \cdots$

$\Rightarrow \quad \theta = \frac{\phi}{n} + \frac{2\pi k}{n}$ $\leftarrow$ k and $k+n$ rep same angle.

Choose representative $0 \le k \le n-1$

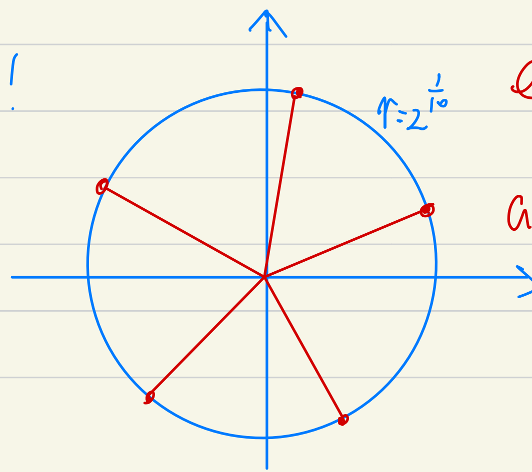
Ex. 5^{th} root of $1+i$: $z^5 = 1+i = \sqrt{2}\, e^{i\frac{\pi}{4}} \Rightarrow z = 2^{\frac{1}{10}} \cdot e^{i\cdot\theta}$.

k	$\cdots -2$	-1	0	1	2	3	4	5	$\cdots$
$\theta = \frac{\pi}{20} + \frac{2\pi\cdot k}{5}$	$-\frac{15\pi}{20}$	$-\frac{7\pi}{20}$	$\frac{\pi}{20}$	$\frac{9\pi}{20}$	$\frac{17\pi}{20}$	$\frac{25\pi}{20}$	$\frac{33\pi}{20}$	$\frac{41\pi}{20}$	$\cdots$

2π

5 different roots !

Geometrically :



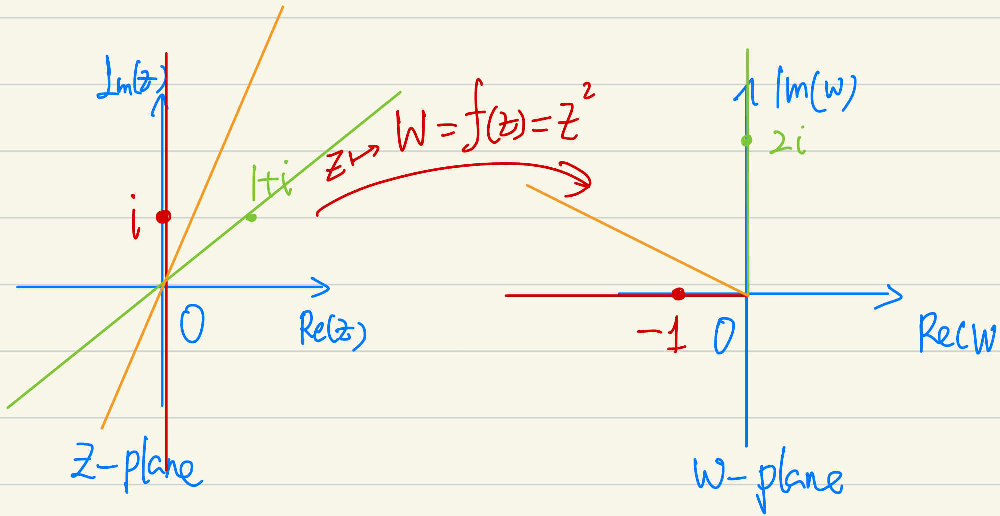
evenly spaced (increment $\frac{2\pi}{5}$)

around circle (radius $2^{\frac{1}{10}}$)

§ Complex Functions / Mappings

$$f: \mathbb{C} \longrightarrow \mathbb{C}$$
$$z \rightsquigarrow w = f(z)$$

- **Geometrically**, Visualize using 2 Complex planes.



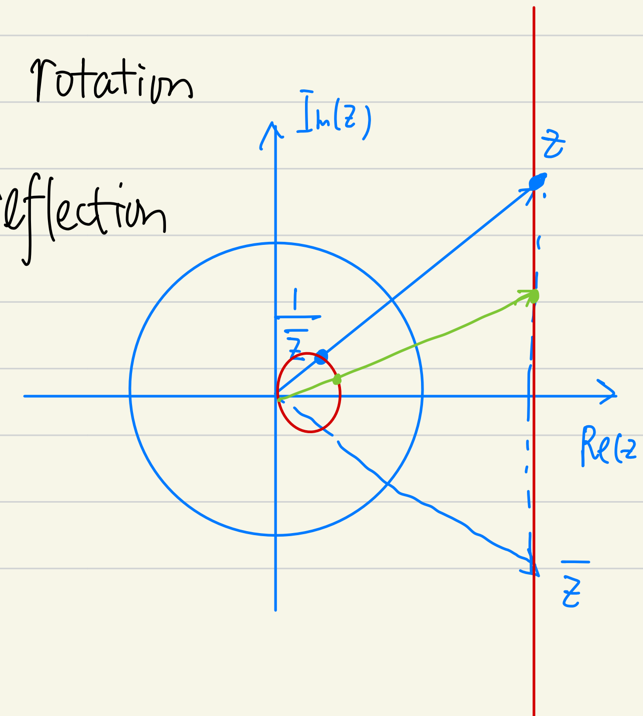
Sometimes, can represent map in 1 complex plane as <u>motions</u>.

- $f(z) = z + z_0 \qquad z_0 \in \mathbb{C}$ translation
- $f(z) = e^{i\theta_0} \cdot z \qquad \theta_0 \in \mathbb{R}$ rotation
- $f(z) = \bar{z}$. reflection
- $f(z) = \frac{1}{\bar{z}} = \overline{\left(\frac{1}{z}\right)}$ inversion

$re^{i\theta} \rightsquigarrow r^{-1}e^{i\theta}$

Domain of definition: $\mathbb{C} - \{0\}$

Range. $\mathbb{C} - \{0\}$



• Algebraically, can rep a complex f^n by a pair of real 2-variable functions

$$f: \underset{\mathbb{R}^2}{\underset{\cong}{\mathbb{C}}} \longrightarrow \underset{\mathbb{R}^2}{\underset{\cong}{\mathbb{C}}}$$

Suppose $w = \boxed{f(z) = u(x,y) + i\,v(x,y)}$

$$\underset{u+iv}{\overset{\|}{w}} = f(x+iy)$$

ex: $f(z) = z^2$. $\quad f(x+iy) = \underbrace{x^2-y^2}_{u(x,y)} + \underbrace{2xy}_{v(x,y)} \cdot i$

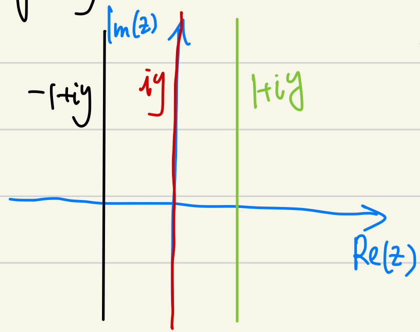
§ Elementary Functions.

(1) Exponential Function:
$$e^z := e^{x+iy} = e^x \cdot e^{iy}$$
$$= e^x \cdot (\cos y + i \sin y)$$

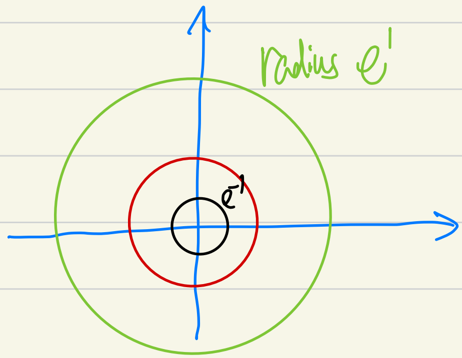
Note:
$$e^{z_1} \cdot e^{z_2} = e^{x_1} \cdot e^{iy_1} \cdot e^{x_2} \cdot e^{iy_2}$$
$$= e^{x_1+x_2} \cdot e^{i(y_1+y_2)}$$
$$= e^{z_1+z_2}$$

"additive property" still holds!

Geometrically,



$z \mapsto W = e^z$



(2) Trigonometric Function $\sin z$, $\cos z$.

Note: $\sin x$, $\cos x$, $x \in \mathbb{R}$ originates from geometry.

Algebraically, $e^{ix} = \cos x + i\sin x$, $e^{-ix} = \cos x - i\sin x$

$$\Rightarrow \sin x = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

Def: $\forall z \in \mathbb{C}$, $\boxed{\sin z := \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}}$

Prop:
- $\sin(-z) = -\sin z$. $\cos(-z) = \cos z$
- $\sin^2 z + \cos^2 z = 1$, $\sin(z_1 + z_2) = \sin z_1 \cdot \cos z_2 + \sin z_2 \cdot \cos z_1$ ...

(Ex.)

(3) Hyperbolic Functions.

· Originated from "hyperbolic geometry"; very different from trig. function with real variables.

$$\boxed{\sinh z := \frac{e^{z}-e^{-z}}{2} \qquad \cosh z := \frac{e^{z}+e^{-z}}{2}.}$$

· $\sinh z = -i\sin(iz)$ $\qquad$ $\cosh z = \cos(iz)$

· $\cosh^2 z - \sinh^2 z = 1$.

(4) Logarithmic Function : $\log(z)$ "inverse of e^z".

(Recall: $e^{\ln x} = x \quad \forall x \in \mathbb{R}$)

Want: $e^w = z$. $w := \log(z)$

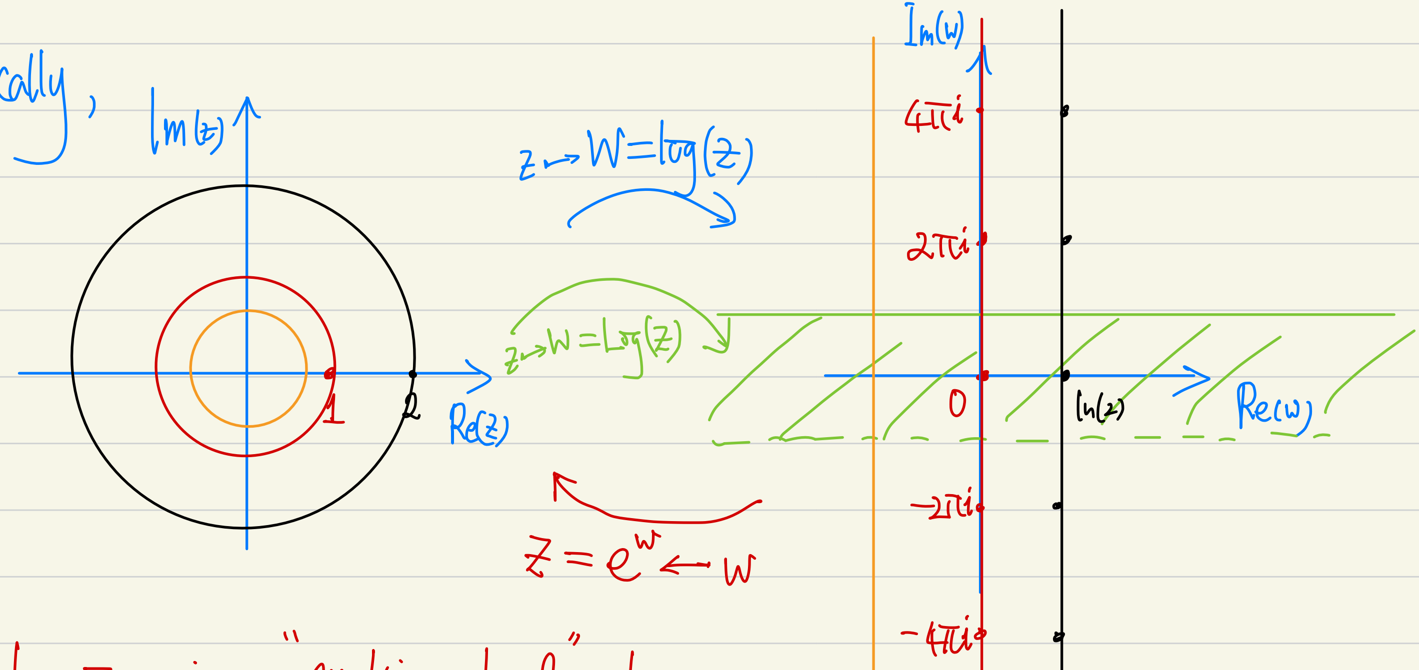
$w = u + iv$. $\Rightarrow e^w = e^u \cdot e^{i \cdot v} = z = r \cdot e^{i\theta}$

$\Rightarrow e^u = r$, $\underbrace{v = \theta + 2k\pi}_{v = \arg(z)}$. $k \in \mathbb{Z}$.

$\Leftrightarrow u = \ln r = \ln|z|$

$\Leftrightarrow \boxed{\log z = \ln|z| + i \cdot \arg(z)} \quad z \in \mathbb{C} - \{0\}$

Geometrically,



Note: $\log z$ is "multi-valued"!

e.g. $\log(1) = 0 + 2k\pi \cdot i$

$\log(2) = \ln 2 + 2k\pi \cdot i$ ...

- **One solution:** Define principal value of $\log z$

$$\operatorname{Log} z := \ln|z| + i\cdot \operatorname{Arg}(z), \text{ where } -\pi < \operatorname{Arg}(z) \le \pi$$

Recall that $\ln(xy) = \ln x + \ln y \quad \forall x, y \in \mathbb{R}$.

However, this is not always true for $\operatorname{Log}(z)$

e.g. $z_0 = e^{i\frac{2\pi}{3}} \Rightarrow \operatorname{Log} z_0 = \frac{2\pi}{3}\cdot i$

$z_0^2 = e^{i\frac{4\pi}{3}} \Rightarrow \operatorname{Log}(z_0^2) = \left(\frac{4\pi}{3} - 2\pi\right)\cdot i = -\frac{2\pi}{3}\cdot i$

$$\operatorname{Log}(z_0^2) \neq 2 \operatorname{Log} z_0 \quad !$$

- **Alternative Solutions:**
  - Branches of functions (e.g. for the branch $0 \le \arg(z) < 2\pi$, $\log(z_0^2) = 2\log z_0$)
  - Riemann surfaces

(5) Power Function. z^c. $c \in \mathbb{C}$.

define $z^c := e^{c \cdot \log(z)}$ $\qquad z \in \mathbb{C} - \{0\}$

$\quad \| $
$e^{\log z}$

Again, multi-valued! (unless $c \in \mathbb{Z}$) Principal-Valued: P.V. $z^c := e^{c \cdot \mathrm{Log}(z)}$

eg $z = c = i$: $\quad i^i = e^{i \cdot \log i}$

$\log(i) = \underbrace{\ln(1)}_{0} + \left(\frac{\pi}{2} + 2k\pi\right) i \qquad \mathrm{Log}(i)$

$\Rightarrow e^{i \cdot \log(i)} = e^{-(2k+\frac{1}{2}) \cdot \pi}$ real numbers! P.V. $i^i = e^{-\frac{1}{2}\pi}$.

(b) Inverse Trigonometric $\sin^{-1} z$, $\cos^{-1} z$

Remember $\sin^{-1} x$ has domain of definition $[-1, 1]$.

For complex function $w = \sin^{-1} z$

$$\Rightarrow \quad z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}$$

$$\Rightarrow \quad (e^{iw})^2 - 2iz \cdot (e^{iw}) - 1 = 0$$

(Solving quadratic eq.) $$\Rightarrow \quad e^{iw} = iz + (1-z^2)^{1/2}$$

$$\Rightarrow \quad iw = \log\left(iz + (1-z^2)^{1/2}\right)$$

Hence, $\boxed{\sin^{-1} z = -i \log\left(iz + (1-z^2)^{1/2}\right)}$

Note: $iz + (1-z^2)^{1/2} \neq 0$. so domain of def of $\sin^{-1}$ = range of $\sin = \mathbb{C}$